MOTION OF A VISCOPLASTIC LIQUID IN A POROUS INHOMOGENEOUS MEDIUM

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ABSTRACT: Equations of motion are derived for a viscoplastic liquid in a nonuniform medium of type 2 (piecewise uniform) or type 3 (with a variable filtration coefficient) [1] on the assumption that the motion is of steady-state type. Solutions are presented for a parallel flow and a flow with axial symmetry.

The motion of a viscoplastic liquid in a porous homogeneous medium has already been considered [1], and it has been shown that the flow rate under engineering structures is finite for a porous medium of infinite size. Equations have been deduced [2] for the motion of a viscoplastic liquid in an inhomogeneous medium, and the conclusions of [1] have been confirmed by experiment [3].

Here I consider the steady-state motion of an incompressible viscoplastic liquid in an inhomogeneous medium of type 2 or type 3. It is assumed that there are no stagnant zones in the flow region. Examples are given of this type of motion.

1. The basic equation extending D'Arcy's law to a viscoplastic fluid is [1]

$$\mathbf{v} = -k\left(1 - \frac{K^*}{|\operatorname{grad} H|}\right)\operatorname{grad} H, \qquad H = \frac{p}{\rho g} + z. \quad (1.1)$$

Here z is the vertical coordinate. We assume that everywhere in the porous medium $|\operatorname{grad} H| > K^*$.

The inhomogeneity of a porous medium is reflected in the filtration coefficient k, whereas here the inhomogeneity is characterized also by variation in the initial gradient K^*. Hence there can be more complicated cases of inhomogeneous media here.

2. Consider a medium of type 2, i.e., piecewise homogeneous. Here the region filled by the porous medium may be divided into n subregions D_j ($j = 1, \ldots, n$), in each of which k and K^* are constant.

All quantities in region D_j are written with subscript j; then in any region $\operatorname{div} \mathbf{v}_j = 0$, and then, from (1.1),

$$\left(1 - \frac{K_j^*}{|\operatorname{grad} H_j|}\right)\Delta H_j + \frac{K_j^* \operatorname{grad} H_j \cdot \operatorname{grad} |\operatorname{grad} H_j|^2}{2|\operatorname{grad} H_j|^3} = 0 \text{ in } D_j, \quad (2.1)$$

or

$$\Delta H_j = \frac{K_j^*}{|\operatorname{grad} H_j|}\left(\Delta H_j - \frac{\operatorname{grad} H_j \cdot \operatorname{grad} |\operatorname{grad} H_j|^2}{2|\operatorname{grad} H_j|^2}\right) \text{in } D_j. \quad (2.2)$$

These relations coincide with (3.10) of [1].

Equations (2.1) must be solved with allowance for the conditions at the boundary of D and for the conditions at the boundary S_{ij} common to regions D_i and D_j ($i \neq j$). The first condition requires that the pressure be continuous at S_{ij},

$$H_i = H_j \qquad \text{at } S_{ij}. \quad (2.3)$$

The second condition implies continuity in the normal velocity at S_{ij},

$$k_i\left(1 - \frac{K_i^*}{|\operatorname{grad} H_i|}\right)\frac{\partial H_i}{\partial n} = k_j\left(1 - \frac{K_j^*}{|\operatorname{grad} H_j|}\right)\frac{\partial H_j}{\partial n}. \quad (2.4)$$

Conditions (2.3) and (2.4) must be used with the small-perturbation method, where H_j is put in the form $H_{j0} + H_j^*$, in which H_{j0} corresponds to an inhomogeneous medium whose initial gradient is zero.

If we assume that $|H_j^*|$ and $|\operatorname{grad} H_j^*|$ are small relative to $|H_{j0}|$ and $|\operatorname{grad} H_{j0}|$ respectively, then we can neglect terms of order K_j^{*n} ($n \geq 2$) in the conditions and

$$H_i^* = H_j^*, \quad k_i \frac{\partial H_i^*}{\partial n} - k_j \frac{\partial H_j^*}{\partial n} = \left(\frac{K_i^*}{|\operatorname{grad} H_{i0}|} - \frac{K_j^*}{|\operatorname{grad} H_{i0}|}\right) k_i \frac{\partial H_{i0}}{\partial n} \text{ at } S_{ij}. \quad (2.5)$$

Function H_j^* satisfies equations readily deduced from (2.1).

If k varies from point to point, the equations of continuity give

$$\left(1 - \frac{K^*}{|\operatorname{grad} H|}\right)(\operatorname{grad} k \cdot \operatorname{grad} H + k\Delta H) + k\left(K^* \frac{\operatorname{grad} |\operatorname{grad} H|^2}{2|\operatorname{grad} H|^3} - \operatorname{grad} K^*\right)\frac{\operatorname{grad} H}{|\operatorname{grad} H|} = 0. \quad (2.6)$$

$K^* = 0$ gives us an equation describing the motion of an ordinary viscous fluid in an inhomogeneous medium of type 3; we get (2.2) if k and K^* are constant.

If the inhomogeneity is of type 2 for k and of type 3 for K^*, the medium may be said to be of type 2-3; similarly, it is of type 3-2 if it is of type 3 for k and type 2 for K^*.

We now consider parallel motion and motion with axial symmetry in a horizontal plate.

3. Let the x axis lie along the flow. We assume that the boundaries of the layer are $x = 0$ and $x = L$, and also that

$$H(0) = H_0 > H(L) = H^\circ.$$

a) Medium of type 2. Let $x = l$ ($0 < l < L$) be the equation of the boundary between two different media; then (1.1) may be written

$$u_j = -k_j\left(\frac{dH_j}{dx} + K_j^*\right)$$

$$(j = 1,\ 0 < x < l;\ j = 2,\ l < x < L) \quad (3.1)$$

with

$$-\frac{dH_j}{dx} > K_j^* \quad (j = 1, 2). \quad (3.2)$$

The equation of continuity implies that $d^2H_j/dx^2 = 0$, from which

$$H_j = A_j x + B_j, \quad (3.3)$$

in which A_j and B_j are constants and $-A_j > K_j^*$. The boundary conditions for $x = 0$ and $x = L$ give

$$H_0 = B_1, \qquad H^\circ = A_2 L + B_2. \quad (3.4)$$

From the conditions at the interfaces we have

$$A_1 l + B_1 = A_2 l + B_2, \qquad k_1(A_1 + K_1^*) = k_2(A_2 + K_2^*). \quad (3.5)$$

Then we get

$$A_1 = -\frac{k_2(H_0 - H^\circ) + (k_1K_1^* - k_2K_2^*)(L - l)}{k_1(L - l) + k_2 l},$$

$$A_2 = -\frac{k_1(H_0 - H^\circ) + (k_2K_2^* - k_1K_1^*)\,l}{k_1(L - l) + k_2 l}, \quad (3.6)$$

$$B_1 = H_0, \quad B_2 = \frac{H^\circ l(k_2 - k_1) + H_0Lk_1 + (k_2K_2^* - k_1K_1^*)\,lL}{k_1(L - l) + k_2 l}.$$

We put $U^{-1} = k_1(L - l) + k_2 l$ to get

$$H_1 = -[k_2(H_0 - H^\circ) + (k_1K_1^* - k_2K_2^*)(L - l)]U_x + H_0,$$
$$H_2 = [k_1(H^\circ - H_0)(x - l) + (k_1K_1^* - k_2K_2^*)l(x - L) + k_2H^\circ l]U. \quad (3.7)$$

The velocity u is defined by

$$u = k_1k_2 \frac{H_0 - H^\circ - K_2^*(L - l) - K_1^* l}{k_1(L - l) + k_2 l}. \quad (3.8)$$

The pressure distribution in a homogeneous medium is not dependent on the initial gradient and is governed by the flow speed. In the present case K_1^* and K_2^* are the result of the pressure and velocity distributions.

b) Medium of type 3. The equation of motion is

$$u = -k\left(\frac{dH}{dx} + K^*\right) \qquad (0 < x < L),$$

and (2.6) is written as

$$k\left(\frac{d^2H}{dx^2} + \frac{dK^*}{dx}\right) + \left(\frac{dH}{dx} + K^*\right)\frac{dk}{dx} = 0.$$

But $k \neq 0$, so we get a second-order equation with variable coefficients,

$$\frac{d^2H}{dx^2} + \frac{d\ln k}{dx}\frac{dH}{dx} + K^*\frac{d\ln k}{dx} + \frac{dK^*}{dx} = 0. \quad (3.9)$$

In the simple case where $k = k_0e^{Cx}$, $K^* = \text{const}$, Eq. (3.9) becomes

$$\frac{d^2H}{dx^2} + C\frac{dH}{dx} + CK^* = 0. \quad (3.10)$$

The general solution is written as

$$H = Ae^{-Cx} - K^*x + B.$$

The boundary conditions give

$$H = [(H_0 - H^\circ - K^*L)e^{-Cx} + H^\circ - H_0e^{-CL} + K^*L](1 - e^{-CL})^{-1} - K^*x. \quad (3.11)$$

The velocity in this case must be

$$u = k_0C\left(\frac{H_0 - H^\circ}{L} - K^*\right)(1 - e^{-CL})^{-1}. \quad (3.12)$$

This expression is clearly always positive.

The pressure and velocity distributions are dependent on K^*. If the parameters of the medium may be represented as linear functions,

$$k = a + bx, \qquad K^* = A^* + B^*x, \quad (3.13)$$

Eq. (3.9) becomes

$$\frac{d^2H}{dx^2} + \frac{1}{\alpha + x}\frac{dH}{dx} + \frac{A + Bx}{\alpha + x} = 0, \quad (3.14)$$

in which

$$\alpha = ab^{-1}, \quad A = A^* + \alpha B^*, \quad B = 2B^*. \quad (3.15)$$

The solution to this equation is sought as

$$H = M\ln(\alpha + x) + N - A[x - \alpha\ln(\alpha + x)] + {}^1/_2B[{}^1/_2(2\alpha x - x^2) - \alpha^2\ln(\alpha + x)]. \quad (3.16)$$

The boundary conditions give us M and N as

$$M = \left[2(H^\circ - H_0) + 2A[L - \alpha\ln(1 + L/\alpha)] + B[\alpha^2\ln(1 + L/\alpha) - \alpha L - {}^1/_2L^2]\right]\cdot\left[2\ln(1 + L/\alpha)\right]^{-1},$$
$$N = H_0 + \frac{\ln\alpha}{\ln(1 + L/\alpha)}\left[H^\circ - H_0 - AL + BL\frac{2\alpha - L}{4}\right], \quad (3.17)$$

and the velocity is

$$u = -(aA^* + bM), \quad (3.18)$$

so the solution for the corresponding homogeneous medium is found for $B^* \to 0$ and $b \to 0$.

4. For motion with central symmetry we denote the boundaries of the porous medium by $r = r_0$ and $r = r^0$, at which H takes the values H_0 and H^0 respectively; we assume that $H_0 > H^0$. Let q be the flow per unit thickness.

a) Medium of the second type. Let $r = R$ ($r_0 < R < r^0$) be the equation of the common boundary of the two media, whose filtration coefficients are respectively k_1 and k_2. Then the velocity u_j may be written as

$$u_j = -k_j\left(\frac{dH_j}{dr} + K^*_j\right) \qquad \left(\begin{matrix} j = 1, & r_0 < r < R \\ j = 2, & R < r < r^\circ \end{matrix}\right). \quad (4.1)$$

The equation of continuity is written as

$$2\pi ru_j = q, \quad (4.2)$$

in which r and j have the values as in (4.1).

From (4.2) we have that H_j satisfies

$$\frac{dH_j}{dr} = -\frac{q}{2\pi k_j r} - K_j^*. \quad (4.3)$$

Then

$$H_j = -\frac{q}{2\pi k_j}\ln r - K_j^*r + M_j.$$

If H_0 and H^0 are given, we have to determine the three constants M_1, M_2, and q, which may be derived from (2.3) with the boundary conditions at $r = r_0$ and $r = r^0$; here (2.4) is obeyed automatically. The equations for the constants are

$$H_0 = -\frac{q}{2\pi k_1}\ln r_0 - K_1^*r_0 - M_1,$$
$$H^\circ = -\frac{q}{2\pi k_2}\ln r^\circ - K_2^*r^\circ + M_2,$$
$$-\frac{q}{2\pi k_1}\ln R - K_1^*R + M_1 = -\frac{q}{2\pi k_2}\ln R - K_2^*R + M_2.$$

The flow rate is given by

$$q = 2\pi\frac{H_0 - H^\circ - K_1^*(R - r_0) - K_2^*(r^\circ - R)}{K_2^{-1}\ln(r^\circ/R) - K_1^{-1}\ln(r_0/R)}, \quad (4.4)$$

which is always positive. For H_1 and H_2 we have

$$H_1 = H_0 - \frac{q}{2\pi k_1}\ln\frac{r}{r_0} - K_1^*(r - r_0) \qquad (r_0 < r < R),$$
$$H_2 = H_\alpha - \frac{q}{2\pi k_2}\ln\frac{r}{r^\circ} + K_2^*(r^\circ - r) \qquad (R < r < r^\circ), \quad (4.5)$$

in which q is defined by (4.4).

b) Medium of the third type. Here H satisfies

$$\frac{dH}{dr} = -\frac{q}{2\pi kr} - K^*.$$

If $k = k(r)$ and $K^* = K^*(r)$ are linear functions,

$$k = a + br, \quad K^* = A + Br, \tag{4.6}$$

function H will be

$$H = -\frac{q}{2\pi a}\ln\frac{r}{r + ab^{-1}} - \left(Ar + B\frac{r^2}{2}\right) + M.$$

The boundary conditions give the flow as

$$q = \pi a \frac{2(H_0 - H^\circ) - 2A(r^\circ - r_0) - B(r^{\circ 2} - r_0^2)}{\ln[r^\circ(a + br_0)] - \ln[r_0(a + br^\circ)]}. \tag{4.7}$$

For H we have

$$H = H^\circ - \frac{q}{2\pi a}\ln\frac{r(a + br^\circ)}{r^\circ(a + br)} - A(r - r^\circ) - \frac{B}{2}(r^2 - r^{\circ 2}). \tag{4.8}$$

Formulas (3.7), (3.8), (3.16), (3.18), (4.4), (4.5), (4.7), (4.8) generalize the expressions for the underground hydrodynamics of a simple viscous liquid to the case of a viscoplastic medium.

More complex inhomogeneous media, which may be called mixed, may be discussed for one-dimensional motion or for motion with central symmetry, as for media of types 2–3 or 3–2, or combinations of these.

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